One equivalent of Schure's Inequality.
https://www.linkedin.com/feed/update/urn:li:activity:6528111651877720064
Let x,y,z be three non-negative real numbers such that $xy+yz+zx+xyz=4$.
Prove that: $x+y+z \geq xy+yz+zx$. When does equality occur?
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Noting that the inequality obviously holds if at least one of the x,y,z equal to zero..
(indeed, let $z=0$ then $xy+yz+zx+xyz=4$ becomes $xy=4$ and since
$x+y \geq 2\sqrt{xy}=4$ then $x+y+z \geq xy+yz+zx$) we assume further that $x,y,z>0$.

Lemma.
All positive solutions of equation $xy+yz+zx+xyz=4$ can be represented in the form

$x=\dfrac{2a}{1-a},\ y=\dfrac{2b}{1-b},\ z=\dfrac{2c}{1-c}$ where $a,b,c>0$ and $a+b+c=1$.

Proof.
First note that $x+2=\dfrac{2}{1-a} \Leftrightarrow 1-a=\dfrac{2}{x+2} \Leftrightarrow a=1-\dfrac{2}{x+2}=\dfrac{x}{x+2}$ and, similarly, we obtain $b=\dfrac{y}{y+2}, c=\dfrac{z}{z+2}$. Also note that $a+b+c=1 \Leftrightarrow \sum(1-a)=2 \Leftrightarrow$
$\sum\dfrac{2}{x+2}=2 \Leftrightarrow xy+yz+zx+xyz=4$.
Then by **Lemma** $x+y+z \geq xy+yz+zx \Leftrightarrow \sum\dfrac{2a}{1-a} \geq \sum\dfrac{2b}{1-b}\cdot\dfrac{2c}{1-c} \Leftrightarrow$
$\sum\dfrac{a}{1-a} \geq 2\sum\dfrac{ab}{(1-a)(1-b)} \Leftrightarrow \sum a(1-b)(1-c) \geq 2\sum ab(1-c) \Leftrightarrow$
$\sum a(a+bc) \geq 2\sum ab(1-c) \Leftrightarrow a^2+b^2+c^2+3abc \geq 2(ab+bc+ca)-6abc \Leftrightarrow$
$9abc \geq 4(ab+bc+ca)-1,$ where latter inequality is Schur's Inequality
$\sum a(a-b)(a-c) \geq 0$ normalized by $a+b+c=1$.
(in homogeneous form $9abc \geq 4(a+b+c)(ab+bc+ca)-(a+b+c)^3$).
Since for $a,b,c>0$ equality in Schure's Inequality occurs iff $a=b=c$ then
in case $x,y,z>0$ original inequality equality occurs iff $x=y=z$,
that is iff $x=y=z=1$ (because the constraint becomes equation
$3x^2+x^3=4$ which in positive x has only one solution $x=1$).
If we allow zero values for x,y,z we have more cases of equality, namely
if $x=y=z=0$ or if $(x,y,z) \in \{(2,2,0),(2,0,2),(0,2,2)\}$.